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ON THE INTEGRABLE SOLUTIONS OF PRODUCT INTEGRO DIFFERENTIAL EQUATIONS IN DIRECT SUM SPACES

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Abstract

In this paper, we consider a product of quasi-differential expressions $\tau_1, \tau_2, \ldots, \tau_n$ each of order n with complex coefficients and their formal adjoints $\tau_1^+, \tau_2^+, \ldots, \tau_n^+$ on $[0, b)$, respectively. We show in the direct sum spaces $L_w^2(I_p)$, $p = 1, 2, \ldots, N$ of functions defined on each of the separate intervals in the case of one singular end-points and under suitable conditions on the function F that all solutions of product integro differential equations $\left[\prod_{j=1}^n \tau_j - \lambda I\right] y(t) = wF$ are bounded and L_w^2-bounded on $[0, b)$.

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1. Introduction

The problem that all solutions of a perturbed linear differential equation belong to $L_w^2(0, \infty)$ assuming the fact that all solutions of the unperturbed equation possess the same property considered by Wong and Zettl [1-3]. In [4] and [6], Ibrahim extends their results for a general quasi-differential expression τ of arbitrary order n with complex coefficients, and considered the property of boundedness of solutions of a general quasi-integro differential equation

$$\tau[y] - \lambda wy = wf(t, y), \quad (\lambda \in \mathbb{C}) \text{ on } [0, b), \tag{1.1}$$

$f(t, y)$ satisfies

$$|f(t, y)| \leq k(t) + h(t)|y(t)|^{\sigma}, \quad t \in [0, b) \text{ for some } \sigma \in [0, 1],$$

provided that all solutions of the equations

$$(\tau - \lambda I)u = 0 \text{ and } (\tau^{+} - \overline{\lambda} I)v = 0 \quad (\lambda \in \mathbb{C}), \tag{1.2}$$

and their quasi-derivatives are in $L_w^2(0, b)$.

Our objective in this paper is to extend the results in [1], [2], and [4-9] to more general class of product quasi-integro differential equations in the form

$$\left[\prod_{j=1}^{n} \tau_j - \lambda I\right] y(t) = wF\left(t, y^{[0]}, y^{[1]}, \ldots, y^{[n^2N-1]}\right) \text{ on } [0, b), \tag{1.3}$$

$F\left(t, y^{[0]}, y^{[1]}, \ldots, y^{[n^2N-1]}\right)$ satisfies

$$\left|F\left(t, y^{[0]}, y^{[1]}, \ldots, y^{[n^2N-1]}\right)\right| \leq k(t) + \sum_{i=0}^{n^2N-1} h_i(t)|y^{[i]}(t)|^{\sigma}, \quad t \in [0, b),$$

for some $\sigma \in [0, 1]$, $0 < b \leq \infty$ in direct sum spaces $L_w^2[0, b_p)$, $p = 1, 2, \ldots, N$ of functions defined on each of the separate intervals with the case of one singular end-points. Also, we prove under suitable conditions on the function F that, if all solutions of the product equation $\left[\prod_{j=1}^{n} \tau_j - \lambda I\right] u = 0$ and its adjoint $\left[\prod_{j=1}^{n} \tau_j^{+} - \overline{\lambda} I\right] v = 0$ and their quasi-derivatives belong to $L_w^2(0, b)$, then all solutions of (1.3) also belong to $L_w^2(0, b)$, where τ_j^{+} is the formal adjoint of τ_j, $j = 1, 2, \ldots, n$.

We deal throughout this paper with a quasi-differential expression τ_j each of arbitrary order n defined by Shin-Zettl matrices (see [6] and [9-11]) on the interval $I = [0, b)$. The left-hand end-point of I is assumed to be regular but the right-hand end-point may be regular or singular.

2. Notation and Preliminaries

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$, respectively, and $N(T)$ will denote its null space. The nullity of T, written $nul\,(T)$, is the dimension of $N(T)$ and the deficiency of T, written $def(T)$, is the co-dimension of $R(T)$ in H; thus if T is densely defined and $R(T)$ is closed , then $def(T) = nul(T^{*})$. The Fredholm domain of T is (in the notation of [13]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda \in \mathbb{C}$, which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in T. Thus $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The index of $(T - \lambda I)$ is the number $ind(T - \lambda I) = nul(T - \lambda I) - def(T - \lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely defined operators A and B acting in a Hilbert space H are said to form an adjoint pair if $A \subset B^{*}$ and, consequently, $B \subset A^{*}$; equivalently, $(Ax, y) = (x, By)$ for all $x \in D(A)$ and $y \in D(B)$, where $(.,\, .)$ denotes the inner-product on H.

Definition 2.1. The field of regularity $\prod(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which, there exists a positive constant $K(\lambda)$ such that

$$\|(A - \lambda I)x\| \geq K(\lambda)\|x\| \text{ for all } x \in D(A), \tag{2.1}$$

or, equivalently, on using the "closed graph theorem", $nul(A - \lambda I) = 0$ and $R(A - \lambda I)$ is closed.

The joint field of regularity $\prod(A, B)$ of A and B is the set of $\lambda \in \mathbb{C}$, which are such that $\lambda \in \prod(A), \overline{\lambda} \in \prod(B)$ and both $def(A - \lambda I)$ and $def(B - \overline{\lambda} I)$ are finite. An adjoint pair of A and B is said to be compatible, if $\prod(A, B) \neq \phi$.

Given two operators A and B, both acting in a Hilbert space H, we wish to consider the product operator AB. This is defined as follows:

$$D(AB) = \{x \in D(B) \mid Bx \in D(A)\} \quad \text{and} \quad (AB)x = A(Bx) \text{ for all } x \in D(AB). \tag{2.2}$$

It may happen in general that $D(AB)$ contains only the null element of H. However, in the case of many differential operators, the domains of the product will be dense in H.

The next result gives conditions under which the deficiency of a product is the sum of the deficiencies of the factors.

Lemma 2.2 (cf. [14, Theorem A]). *Let A and B be closed operators with dense domains in a Hilbert space H. Suppose that $\lambda = 0$ is a regular type point for both operators and $def(A)$ and $def(B)$ are finite. Then AB is a closed operator with dense domain, has $\lambda = 0$ as a regular type point and*

$$def(AB) = def(A) + def(B). \tag{2.3}$$

Proof. The proof is similar to that in [5], [7], [12], and [16].

Evidently, Lemma 2.2 extends to the product of any finite number of operators $A_1, A_2, \ldots, A_n$.

3. Quasi-Differential Expressions in Direct Sum Spaces

The quasi-differential expressions are defined in terms of a Shin-Zettl matrix F_p on an interval I_p. The set $Z_n(I_p)$ of Shin-Zettl matrices on I_p consists of $n \times n$-matrices $F_p = \{f_{rs}^p\}$, $p = 1, 2, \ldots, N$ whose entries are complex-valued functions on I_p, which satisfy the following conditions:

$$
\begin{aligned}
& f_{rs}^p \in L_{loc}^2(I_p), \quad 1 \leq r, s \leq n, \quad n \geq 2, \\
& f_{r,r+1}^p \neq 0, \qquad \text{a.e., on } I_p, \quad 1 \leq r \leq n-1, \qquad (3.1) \\
& f_{rs}^p = 0, \qquad \text{a.e., on } I_p, \quad 2 \leq r+1 < s \leq n, \quad p = 1, 2, \ldots, N.
\end{aligned}
$$

For $F_p \in Z_n(I_p)$, the quasi-derivatives associated with F_p are defined by

$$
\begin{aligned}
& y^{[0]} := y, \\
& y^{[r]} := (f_{r,r+1}^p)^{-1}\left\{\left(y^{[r-1]}\right)' - \sum_{s=1}^{r} f_{rs}^p y^{[s-1]}\right\}, \quad 1 \leq r \leq n-1, \qquad (3.2) \\
& y^{[n]} := \left\{\left(y^{[n-1]}\right)' - \sum_{s=1}^{n} f_{ns}^p y^{[s-1]}\right\},
\end{aligned}
$$

where the prime$'$ denotes differentiation.

The quasi-differential expression τ_p associated with F_p is given by

$$
\tau_p[.] := i^n y^{[n]}, \quad n \geq 2, \qquad (3.3)
$$

this being defined on the set

$$V(\tau_p) := \left\{y : y^{[r-1]} \in AC_{loc}(I_p),\ r = 1, 2, \ldots, n\right\}, \quad p = 1, 2, \ldots, N,$$

where $AC_{loc}(I_p)$, denotes the set of functions, which are absolutely continuous on every compact subinterval of I_p.

The formal adjoint τ_p^+ of τ_p is defined by the matrix F_p^+ given by

$$\tau_p^+[.\ ] := i^n y_+^{[n]}, \text{ for all } y \in V(\tau_p^+), \tag{3.4}$$

$$V(\tau_p^+) := \left\{y : y_+^{[r-1]} \in AC_{loc}(I_p),\ r = 1, 2, \ldots, n\right\}, \quad p = 1, 2, \ldots, N,$$

where $y_+^{[r-1]}$, the quasi-derivatives associated with the matrix F_p^+ in $Z_n(I_p)$,

$$F_p^+ = (f_{rs}^p)^+ = (-1)^{r+s+1}\overline{f_{n-s+1,\, n-r+1}^p}, \qquad \text{for each } r \text{ and } s. \tag{3.5}$$

Note that $(f_p^+)^+ = F_p$ and so $(\tau_p^+)^+ = \tau_p$. We refer to [4], [6, 7], [10-13], and [15, 16] for a full account of the above and subsequent results on quasi-differential expressions.

For $u \in V(\tau_p)$, $v \in V(\tau_p^+)$, and $\alpha, \beta \in I_p$, we have Green's formula

$$\int_{a_p}^{b_p} \left\{\bar{v}\tau_p[u] - u\overline{\tau_p^+[v]}\right\} dx = [u, v](b_p) - [u, v](a_p), \quad p = 1, 2, \ldots, N, \tag{3.6}$$

where

$$[u, v](x) = i^n \left(\sum_{r=0}^{n-1} (-1)^{r+s+1} u^{[r]}(x) \overline{v_+^{[n-r-1]}}(x)\right)$$

$$= (-i)^n \left(u, u^{[1]}, \ldots, u^{[n-1]}\right) \times J_{n \times n} \begin{pmatrix} \bar{v} \\ \vdots \\ \bar{v}_+^{[n-1]} \end{pmatrix}(x); \tag{3.7}$$

see [1], [4], [7], [10, Corollary 1], and [15].

Let the interval I_p have end-points $a_p, b_p(-\infty \leq a_p < b_p \leq \infty)$, and let $w_p : I_p \to \mathbb{R}$ be a non-negative weight function with $w_p \in L^1_{loc}(I_p)$ and $w_p > 0$ (for almost all $x \in I_p$). Then $H_p = L^2_{w_p}(I_p)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that $\int_{I_p} w_p |f|^2 < \infty;$ the inner-product is defined by

$$(f, g)_p := \int_{I_p} w_p f(x)\overline{g(x)}dx, \quad (f, g \in L^2_{w_p}(I_p), \; p = 1, 2, \dots, N). \quad (3.8)$$

The equation

$$\tau_p[u] - \lambda w_p u = 0 \quad (\lambda \in \mathbb{C}) \quad \text{on } I_p, \quad p = 1, 2, \dots, N, \quad (3.9)$$

is said to be **regular** at the left end-point $a_p \in \mathbb{R},$ if for all $X \in (a_p, b_p),$

$$a_p \in \mathbb{R}, \quad w_p, f^p_{rs} \in L^1(a_p, X), \quad r, s = 1, 2, \dots, n; \; p = 1, 2, \dots, N.$$

Otherwise (3.9) is said to be **singular** at a_p. If (3.9) is regular at both end-points, then it is said to be regular; in this case, we have

$$a_p, b_p \in \mathbb{R}, \quad w_p, f^p_{rs} \in L^1(a_p, b_p), \quad r, s = 1, 2, \dots, n; \; p = 1, 2, \dots, N.$$

We shall be concerned with the case when a_p is a regular end-point of (3.9), the end-point b_p being allowed to be either regular or singular. Note that, in view of (3.5), an end-point of I_p is regular for (3.9), if and only if it is regular for the equation

$$\tau^+_p[v] - \overline{\lambda} w_p v = 0 \quad (\lambda \in \mathbb{C}) \quad \text{on } I_p, \quad p = 1, 2, \dots, N. \quad (3.10)$$

Note that at a regular end-point a_p, say, $u^{[r-1]}(a_p)\left(v^{[r-1]}_+(a_p)\right)$, $r = 1, 2, \dots, n$ is defined for all $u \in V(\tau_p)\left(v \in V(\tau^+_p)\right)$. Set

$$D(\tau_p) := \{u : u \in V(\tau_p),\ u \text{ and } w_p^{-1}\tau_p[u] \in L^2_{w_p}(a_p, b_p)\},\ p = 1, 2, \dots, N,$$

$$D(\tau_p^+) := \{v : v \in V(\tau_p^+),\ v \text{ and } w_p^{-1}\tau_p^+[v] \in L^2_{w_p}(a_p, b_p)\},\ p = 1, 2, \dots, N. \tag{3.11}$$

The subspaces $D(\tau_p)$ and $D(\tau_p^+)$ of $L^2_{w_p}(a_p, b_p)$ are domains of the so-called maximal operators $T(\tau_p)$ and $T(\tau_p^+)$, respectively, defined by

$$T(\tau_p)u := w_p^{-1}\tau_p[u],\ (u \in D(\tau_p)) \text{ and } T(\tau_p^+)v := w_p^{-1}\tau_p^+[v],\ (v \in D(\tau_p^+)).$$

For the regular problem, the minimal operators $T_0(\tau_p)$ and $T_0(\tau_p^+)$, $p = 1, 2, \dots, N$ are the restrictions of $w_p^{-1}\tau_p[u]$ and $w_p^{-1}\tau_p^+[v]$ to the subspaces

$$D_0(\tau_p) := \{u : u \in D(\tau_p),\ u^{[r-1]}(a_p) = u^{[r-1]}(b_p),\ p = 1, 2, \dots, N\},$$

$$D_0(\tau_p^+) := \{v : v \in D(\tau_p^+),\ v_+^{[r-1]}(a_p) = v_+^{[r-1]}(b_p),\ p = 1, 2, \dots, N\}, \tag{3.12}$$

respectively. The subspaces $D_0(\tau_p)$ and $D_0(\tau_p^+)$ are dense in $L^2_{w_p}(a_p, b_p)$ and $T_0(\tau_p)$ and $T_0(\tau_p^+)$ are closed operators (see [4], [7], [10, Section 3], [12, 13], and [15, 16]).

In the singular problem, we first introduce the operators $T_0'(\tau_p)$ and $T_0'(\tau_p^+)$; $T_0'(\tau_p)$ being the restriction of $w_p^{-1}\tau_p[.]$ to the subspace

$$D_0'(\tau_p) := \{u : u \in D(\tau_p),\ supp\,(u) \subset (a_p, b_p),\ p = 1, 2, \dots, N\}, \tag{3.13}$$

and with $T_0'(\tau_p^+)$ defined similarly. These operators are densely-defined and closable in $L^2_{w_p}(a_p, b_p)$; and we define the minimal operators $T_0(\tau_p)$ and $T_0(\tau_p^+)$ to be their respective closures (see [4], [10], [13], and [15]).

We denote the domains of $T_0(\tau_p)$ and $T_0(\tau_p^+)$ by $D_0(\tau_p)$ and $D_0(\tau_p^+)$, respectively. It can be shown that

$$u \in D_0(\tau_p) \Rightarrow u^{[r-1]}(a_p) = 0, \quad (r = 1, 2, \ldots, n;\ p = 1, 2, \ldots, N),$$

$$v \in D_0(\tau_p^+) \Rightarrow v_+^{[r-1]}(a_p) = 0, \quad (r = 1, 2, \ldots, n;\ p = 1, 2, \ldots, N), \tag{3.14}$$

because we are assuming that a_p is a regular end-point. Moreover, in both regular and singular problems, we have

$$T_0^*(\tau_p) = T(\tau_p^+), \quad T_0^*(\tau_p) = T_0(\tau_p^+), \quad p = 1, 2, \ldots, N; \tag{3.15}$$

see [9, Section 5] in the case when $\tau_p = \tau_p^+$ and compare with treatment in [4] and [13, Section III, 10.3] in general case.

We summarize a few additional properties of $T_0(\tau)$ in the form of a lemma.

Lemma 3.1. *We have*

(i) $[T_0(\tau)]^* = \oplus_{p=1}^N [T_0(\tau_p)]^* = \oplus_{p=1}^N [T(\tau_p^+)],$

$[T_0(\tau^+)]^* = \oplus_{p=1}^N [T_0(\tau_p^+)]^* = \oplus_{p=1}^N [T(\tau_p)].$

In particular,

$$D[T_0(\tau)]^* = D[T(\tau^+)] = \oplus_{p=1}^N [T(\tau_p^+)],$$

$$D[T_0(\tau^+)]^* = D[T(\tau)] = \oplus_{p=1}^N [T(\tau_p)].$$

(ii) $nul[T_0(\tau) - \lambda I] = \sum_{p=1}^N nul[T_0(\tau_p) - \lambda I],$

$nul[T_0(\tau^+) - \bar{\lambda} I] = \sum_{p=1}^N nul[T_0(\tau_p^+) - \bar{\lambda} I].$

(iii) *The deficiency indices of* $T_0(\tau)$ *are given by*

$$def[T_0(\tau) - \lambda I] = \sum_{p=1}^{N} def[T_0(\tau_p) - \lambda I] \text{ for all } \lambda \in \Pi[T_0(\tau_p)],$$

$$def[T_0(\tau^+) - \bar{\lambda} I] = \sum_{p=1}^{N} def[T_0(\tau_p^+) - \bar{\lambda} I] \text{ for all } \lambda \in \Pi[T_0(\tau_p^+)].$$

Proof. Part (a) follows immediately from the definition of $T_0(\tau)$ and from the general definition of an adjoint operator. The other parts are either direct consequences of part (a) or follow immediately from the definitions.

Lemma 3.2. *Let* $T_0(\tau) = \oplus_{p=1}^{N} T_0(\tau_p)$ *be a closed densely-defined operator on H. Then*,

$$\Pi[T_0(\tau)] = \cap_{p=1}^{N} [T_0(\tau_p)].$$

Proof. The proof follows from Lemma 3.1 and since $R[T_0(\tau) - \lambda I]$ is closed if and only if $R[T_0(\tau_p) - \lambda I]$, $p = 1, 2, \ldots, N$ are closed.

4. The Product Operators in Direct Sum Spaces

The proof of general theorems will be based on the results in this section. We start by listing some properties and results of quasi-differential expressions $\tau_1, \tau_2, \ldots, \tau_n$. For proofs, the reader is referred to [3], [7-10], and [14-18].

$$(\tau_1 + \tau_2)^+ = \tau_1^+ + \tau_2^+,$$

$$(\tau_1 \tau_2)^+ = \tau_2^+ \tau_1^+, \; (\lambda\tau)^+ = \bar{\lambda}\tau^+ \text{ for } \lambda \text{ a complex number.} \tag{4.1}$$

A consequence of Properties (4.1) is that if $\tau^+ = \tau$, then $(P(\tau))^+ = P(\tau^+)$ for P any polynomial with complex coefficients. Also, we note that the leading coefficients of a product is the product of the leading coefficients. Hence, the product of regular differential expressions is regular.

Lemma 4.1 (cf. [12, Theorem 1]). *Suppose* τ_j *is a regular differential expression on the interval* $[a, b]$ *and* $\lambda \in \Pi[T_0(\tau_1\ \tau_2\ \dots\ \tau_n),$ $T_0(\tau_1\ \tau_2\ \dots\ \tau_n)^+]$, *then we have*

(i) *The product operator* $\prod_{j=1}^{n} T_0(\tau_j)$ *is closed, densely-defined, and*

$$def\left[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I\right] = \sum_{j=1}^{n} def[T_0(\tau_j) - \lambda I],$$

$$def\left[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I\right] = \sum_{j=1}^{n} def[T_0(\tau_j^+) - \bar{\lambda} I].$$

(ii) $T_0(\tau_1\ \tau_2\ \dots\ \tau_n) \subseteq \prod_{j=1}^{n}[T_0(\tau_j)]$ *and* $T_0(\tau_1\ \tau_2\ \dots\ \tau_n)^+ \subseteq \prod_{j=1}^{n}$ $[T_0(\tau_j^+)]$.

Note in part (ii) that the containment may be proper, i.e., the operators $T_0(\tau_1\ \tau_2\ \dots\ \tau_n)$ and $\prod_{j=1}^{n}[T_0(\tau_j)]$ are not equal in general. We refer to [6, 7], [15], and [16] for more details.

From Lemmas 3.1 and 4.1, we have the following:

Lemma 4.2. *For* $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$, *we have*

(i) $[\prod_{j=1}^{n} T_0^*(\tau_j)] = \oplus_{p=1}^{N}\left[\prod_{j=1}^{n} T_0^*(\tau_{jp})\right] = \oplus_{p=1}^{N}\left[\prod_{j=1}^{n} T(\tau_{jp}^+)\right],$

$$[\prod_{j=1}^{n} T_0^*(\tau_j^+)] = \oplus_{p=1}^{N}\left[\prod_{j=1}^{n} T_0^*(\tau_{jp}^+)\right] = \oplus_{p=1}^{N}\left[\prod_{j=1}^{n} T(\tau_{jp})\right].$$

(ii) $nul[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I] = \sum_{p=1}^{N} nul\left[\prod_{j=1}^{n} T_0(\tau_{jp}) - \lambda I\right]$

$$= \sum_{p=1}^{N}(\sum_{j=1}^{N} nul[T_0(\tau_{jp}) - \lambda I]),$$

$$nul[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I] = \sum_{p=1}^{N} nul\left[\prod_{j=1}^{n} T_0(\tau_{jp}^+) - \bar{\lambda} I\right]$$

$$= \sum_{p=1}^{N} (\sum_{j=1}^{n} nul\left[T_0(\tau_{jp}^+) - \bar{\lambda} I\right]).$$

(iii) *The deficiency indices of* $\prod_{j=1}^{n} T_0(\tau_j)$ *and* $\prod_{j=1}^{n} T_0(\tau_j^+)$ *are given by*

$$def[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I] = \sum_{p=1}^{N} def\left[\prod_{j=1}^{n} T_0(\tau_{jp}) - \lambda I\right]$$

$$= \sum_{p=1}^{N} (\sum_{j=1}^{n} def\,[T_0(\tau_{jp}) - \lambda I]),$$

$$def[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I] = \sum_{p=1}^{N} def\left[\prod_{j=1}^{n} T_0(\tau_{jp}^+) - \bar{\lambda} I\right]$$

$$= \sum_{p=1}^{N} (\sum_{j=1}^{n} def\left[T_0(\tau_{jp}^+) - \bar{\lambda} I\right]).$$

Lemma 4.3. *For* $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)],\ \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$,

$def\left[\prod_{j=1}^{n}[T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right]$ *is constant, and*

$$0 \le def\left[\prod_{j=1}^{n}[T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right] \le 2n^2 N.$$

In the problem with one singular end-point,

$$n^2 N \le def\left[\prod_{j=1}^{n}[T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right] \le 2n^2 N,$$

for all $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)],\ \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$.

In the regular problem,

$$def\left[\prod_{j=1}^{n}[T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right] = 2n^2 N,$$

for all $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)],\ \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$.

Proof. The proof is similar to that in [4], [6, 7], and [12-16] and therefore omitted.

Lemma 4.4. *Let* $\tau_1, \tau_2, \ldots, \tau_n$ *be a regular differential expressions on* $[0, b)$ *and suppose that* $\lambda \in \Pi[T_0(\tau_1\ \tau_2\ \ldots\ \tau_n),\ T_0(\tau_1\ \tau_2\ \ldots\ \tau_n)^+]$. *Then*

$$[T_0(\tau_1\ \tau_2\ \ldots\ \tau_n)] = \prod_{j=1}^{n}[T_0(\tau_j)], \tag{4.2}$$

if and only if the following partial separation conditions is satisfied:

$\{f \in L_w^2(a, b),\ f^{[s-1]} \in AC_{loc}[0, b),$ *where s is the order of product expression* $(\tau_1\ \tau_2\ \ldots\ \tau_n)$ *and* $(\tau_1\ \tau_2\ \ldots\ \tau_n)^+ f \in L_w^2(0, b)$ *together imply that*

$$\left(\prod_{j=1}^{k}\left(\tau_j^+\right)\right) f \in L_w^2(0, b), \quad k = 1, \ldots, n-1\}. \tag{4.3}$$

Furthermore,

$$T_0(\tau_1\ \tau_2\ \ldots\ \tau_n) = \prod_{j=1}^{n}[T_0(\tau_j)] \text{ and } T_0(\tau_1\ \tau_2\ \ldots\ \tau_n)^+ = \prod_{j=1}^{n}[T_0(\tau_j^+)].$$

If and only if

$$def[T_0(\tau_1\ \tau_2\ \ldots\ \tau_n) - \lambda I] = \sum_{j=1}^{n} def\left[T_0(\tau_j) - \lambda I\right],$$

$$def\left[T_0(\tau_1\ \tau_2\ \ldots\ \tau_n)^+ - \bar{\lambda} I\right] = \sum_{j=1}^{n} def\left[T_0(\tau_j^+) - \bar{\lambda} I\right].$$

We will say that the product $\tau_1, \tau_2, \ldots, \tau_n$ *is partially separated expressions in* $L^2_w(a, b)$ *whenever Property* (4.3) *holds.*

Corollary 4.5 (cf. [5, Corollary 1]). *Let* τ_j *is a regular differential expressions on* $[0, b)$ *for* $j = 1, 2, \ldots, n$. *If all solutions of the differential equation* $(\tau_j - \lambda I)u = 0$ *and* $(\tau_j^+ - \bar{\lambda} I)v = 0$ *on* $[0, b)$ *are in* $L^2_w(0, b)$ *for* $j = 1, 2, \ldots, n$ *and* $\lambda \in \mathbb{C}$; *then all solutions of* $[\prod_{j=1}^n \tau_j - \lambda I]u = 0$ *and* $(\prod_{j=1}^n \tau_j^+ - \bar{\lambda} I)v = 0$ *on* $[0, b)$ *are in* $L^2_w(0, b)$ *for all* $\lambda \in \mathbb{C}$.

Proof. Let $n = n_j =$ order of $\tau_j =$ order of τ_j^+ for $j = 1, 2, \ldots, n$. Then by Lemma 4.1, we have

$$def[\prod_{j=1}^n T_0(\tau_j) - \lambda I] = \sum_{p=1}^N def\left[\prod_{j=1}^n T_0(\tau_{jp}) - \lambda I\right] = n^2,$$

$$def\left[\prod_{j=1}^n T_0(\tau_j^+) - \bar{\lambda} I\right] = \sum_{p=1}^N def\left[\prod_{j=1}^n T_0\left(\tau_{jp}^+\right) - \bar{\lambda} I\right] = n^2.$$

Hence, by Lemma 4.2, we have

$$def[T_0(\tau_1 \ \tau_2 \ \ldots \ \tau_n)^+ - \lambda I] = def\left[\prod_{j=1}^n T_0(\tau_j^+) - \bar{\lambda} I\right] = \sum_{p=1}^N \sum_{j=1}^n n_j$$

$$= n^2 N = \text{order of } (\tau_1 \ \tau_2 \ \ldots \ \tau_n) = \text{order of } (\tau_1 \ \tau_2 \ \ldots \ \tau_n)^+.$$

Thus $def[T_0(\tau_n^+ \ \ldots \ \tau_2^+ \ \tau_1^+) - \lambda I] =$ order of $(\tau_1 \ \tau_2 \ \ldots \ \tau_n)^+$ and consequently, all solutions of the equations $[\prod_{j=1}^n \tau_j - \lambda I]u = 0$ and $(\prod_{j=1}^n \tau_j^+ - \bar{\lambda} I)v = 0$ are in $L^2_w(0, b)$. Repeating this argument with τ_j^+ replaced by τ_j, we conclude that all solutions of $(\prod_{j=1}^n \tau_j^+ - \bar{\lambda} I)v = 0$ are in $L^2_w(0, b)$.

The special case of Corollary 4.6 when $\tau_j = \tau$ for $j = 1, 2, \ldots, n$ and τ is symmetric was established in [2] and [5]. In this case, it is easy to see that the converse also holds. If all solutions of $(\tau^n - \lambda I)u = 0$ are in $L_w^2(0, b)$, then all solutions of $(\tau - \lambda I)u = 0$ must be in $L_w^2(0, b)$. In general, if all solutions of $[(\tau_1 \ \tau_2 \ \ldots \ \tau_n) - \lambda I]u = 0$ are in $L_w^2(0, b)$, then all solutions of $(\tau_n - \lambda I)u = 0$ are in $L_w^2(0, b)$ since these are also solutions of $[(\tau_1 \ \tau_2 \ \ldots \ \tau_n) - \lambda I]u = 0$. If all solutions of the adjoint equation $[(\tau_1 \ \tau_2 \ \ldots \ \tau_n)^+ - \bar{\lambda} I]v = 0$ are also in $L_w^2(0, b)$, then it follows similarly that all solutions of $(\tau_j^+ - \bar{\lambda} I)v = 0$ are in $L_w^2(0, b)$.

Denote by $S(\tau)$ and $S(\tau^+)$ the sets of all solutions of the equations

$$[\prod_{j=1}^{n} \tau_j - \lambda_0 I]u = 0 \text{ and } (\prod_{j=1}^{n} \tau_j^+ - \overline{\lambda_0} I)v = 0, \tag{4.5}$$

respectively, and let $S^r(\tau) = \{ y^{[r]} : [\prod_{j=1}^{n} \tau_j - \lambda_0 I]y = 0, r = 1, 2, \ldots, n^2 N - 1\}$

denote the set of all quasi-derivatives of solutions of $[\prod_{j=1}^{n} \tau_j - \lambda_0 I]u = 0.$

Let $\varphi_k(t, \lambda)$, $k = 1, 2, \ldots, n^2 N$ be the solutions of the homogeneous equation

$$\left[\prod_{j=1}^{n} \tau_j - \lambda I\right]u = 0 \quad (\lambda \in \mathbb{C}), \tag{4.6}$$

satisfying

$$\varphi_j^{[k-1]}(t_0, \lambda) = \delta_{k, r+1} \text{ for all } t_0 \in [a, b),$$

$$j, k = 1, 2, \ldots, n^2 N, \ r = 0, 1, \ldots, n^2 N - 1,$$

for fixed t_0, $a < t_0 < b$. Then $\varphi_j^{[r]}(t, \lambda)$ is continuous in (t, λ) for $a < t < b$, $|\lambda| < \infty$, and for fixed t, it is entire in λ. Let $\varphi_k^{+}(t, \lambda)$, $k = 1, 2, \ldots, n^2N$ denote the solutions of the adjoint homogeneous equation

$$\left[\prod_{j=1}^{n} \tau_j^{+} - \overline{\lambda} I\right] v = 0 \quad (\lambda \in \mathbb{C}), \tag{4.7}$$

satisfying

$$(\varphi_k^{+})^{[r]}(t_0, \lambda) = (-1)^{k+r} \delta_{k, n^2N-r} \text{ for all } t_0 \in [0, b),$$

$$k = 1, 2, \ldots, n^2N, \; r = 0, 1, \ldots, n^2N - 1.$$

Suppose $a < c < b$. By [9], [14], and [15], a solution of the product equation

$$\left[\prod_{j=1}^{n} \tau_j - \lambda I\right] u = wf \quad (\lambda \in \mathbb{C}), \quad f \in L_w^1(0, b), \tag{4.8}$$

satisfying $u^{[r]}(c) = 0$, $r = 0, 1, \ldots, n^2N - 1$ is giving by

$$\varphi(t, \lambda) = \left(\frac{(\lambda - \lambda_0)}{i^{n^2N}}\right) \sum_{j, k=1}^{n^2N} \xi^{jk} \varphi_j(t, \lambda) \int_a^t \overline{\varphi_k^{+}(s, \lambda)} f(s) w(s) ds,$$

where $\varphi_k^{+}(t, \lambda)$ stands for the complex conjugate of $\varphi_k(t, \lambda)$ and for each j, k, ξ^{jk} is constant, which is independent of t, λ (but does depend in general on t_0).

The next lemma is a form of the variation of parameters formula for a general quasi-differential equation is giving by the following lemma:

Lemma 4.6. *Suppose* $f \in L_w^1(0, b)$ *locally integrable function and* $\varphi(t, \lambda)$ *is the solution of the Equation* (4.8) *satisfying*

$$\varphi^{[r]}(t_0, \lambda) = \alpha_{r+1} \text{ for } r = 0, 1, \ldots, n^2N - 1, \quad t_0 \in [0, b).$$

Then

$$\varphi(t, \lambda) = \sum_{j=1}^{n^2 N} \alpha_j(\lambda)\varphi_j(t, \lambda_0) + \left((\lambda - \lambda_0)/i^{n^2 N}\right)$$

$$\times \sum_{j,k=1}^{n^2 N} \xi^{jk}\varphi_j(t, \lambda_0)\int_a^t \overline{\varphi_k^+(s, \lambda_0)} f(s)w(s)ds, \tag{4.9}$$

for some constants $\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_{n^2}(\lambda) \in \mathbb{C}$, *where* $\varphi_j(t, \lambda_0)$ *and* $\varphi_k^+(t, \lambda_0)$, $j, k = 1, 2, \ldots, n^2 N$ *are solutions of the equations in* (4.5), *respectively*, ξ^{jk} *is a constant which is independent of* t.

Proof. The proof is similar to that in [6, 7], [10, 11], and [15, 16]. Lemma 4.6 contain the following lemma as a special case:

Lemma 4.7. *Suppose* $f \in L_w^1(0, b)$ *locally integrable function and* $\varphi(t, \lambda)$ *is the solution of the Equation* (4.8) *satisfying*

$$\varphi^{[r]}(t_0, \lambda) = \alpha_{r+1} \text{ for } r = 0, 1, \ldots, n^2 - 1, \quad t_0 \in [0, b).$$

Then

$$\varphi^{[r]}(t, \lambda) = \sum_{j=1}^{n^2 N} \alpha_j(\lambda)\varphi_j^{[r]}(t, \lambda_0) + \frac{1}{i^{n^2 N}}(\lambda - \lambda_0)$$

$$\times \sum_{j,k=1}^{n^2 N} \xi^{jk}\,\varphi_j^{[r]}(t, \lambda_0)\int_a^t \overline{\varphi_k^+(t, \lambda_0)} f(s)w(s)ds, \tag{4.10}$$

for $r = 0, 1, \ldots, n^2 N - 1$. *We refer to* [3], [9], *and* [14] *for more details.*

Lemma 4.8. *Suppose that for some* $\lambda_0 \in \mathbb{C}$ *all solutions of the equations in* (4.5) *are in* $L_w^2(0, b)$. *Then all solutions of the Equations* (4.6) *and* (4.7) *are in* $L_w^2(0, b)$ *for every complex number* $\lambda \in \mathbb{C}$.

Proof. The proof is similar to that in [16, Lemma 3.3].

Lemma 4.9. *If all solutions of the equation* $[\prod_{j=1}^{n} \tau_j - \lambda_0 w]u = 0$ *are bounded on* $[0, b)$ *and* $\varphi_k^+(t, \lambda_0) \in L_w^1(0, b)$ *for some* $\lambda_0 \in \mathbb{C}$, $k = 1, \ldots, n^2 N$. *Then all solutions of the equation* $[\prod_{j=1}^{n} \tau_j - \lambda w]u = 0$ *are also bounded on* $[0, b)$ *for every complex number* $\lambda \in \mathbb{C}$.

Proof. The proof is similar to that in [16, Lemma 3.3].

Lemma 4.10. *Suppose that for some complex number* $\lambda_0 \in \mathbb{C}$, *all solutions of the equations in* (4.5) *are in* $L_w^2(0, b)$. *Suppose* $f \in L_w^2(0, b)$, *then all solutions of the Equation* (4.8) *are in* $L_w^2(0, b)$ *for all* $\lambda \in \mathbb{C}$.

Proof. Let $\{\varphi_1(t, \lambda), \varphi_2(t, \lambda), \ldots, \varphi_{n^2 N}(t, \lambda)\}$, $\{\varphi_1^+(s, \lambda), \varphi_2^+(s, \lambda), \ldots, \varphi_{n^2 N}^+(s, \lambda)\}$ be two sets of linearly independent solutions of the Equations (4.5). Then for any solutions $\varphi(t, \lambda)$ of the equation $[\prod_{j=1}^{n} \tau_j - \lambda I]\varphi = wf$ $(\lambda \in \mathbb{C})$, which may be written as follows $[\prod_{j=1}^{n} (\tau_j) - \lambda_0 w]\varphi = (\lambda - \lambda_0)wf + wf$ and it follows from (4.9) that

$$\varphi(t, \lambda) = \sum_{j=1}^{n^2 N} \alpha_j(\lambda)\varphi_j(t, \lambda_0) + \frac{1}{i^{n^2 N}} \sum_{j, k=1}^{n^2 N} \xi^{jk} \varphi_j(t, \lambda_0)$$

$$\times \int_a^t \overline{\varphi_k^+(t, \lambda_0)}[(\lambda - \lambda_0)\varphi(s, \lambda) + f(s)]w(s)ds, \tag{4.11}$$

for some constants $\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_{n^2 N}(\lambda) \in \mathbb{C}$. Hence

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n^2 N} \left(|\alpha_j(\lambda)| \, |\varphi_j(t, \lambda_0)|\right) + \sum_{j, k=1}^{n^2 N} \left|\xi^{jk}\right| |\varphi_j(t, \lambda_0)|$$

$$\times \int_a^t \overline{\varphi_k^+(t, \lambda_0)}[|\lambda - \lambda_0| \, |\varphi(s, \lambda)| + |f(s)|]w(s)ds. \tag{4.12}$$

Since $f \in L_w^2(0, b)$ and $\varphi_k^+(., \lambda_0) \in L_w^2(0, b)$ for some $\lambda_0 \in \mathbb{C}$, then $\varphi_k^+(., \lambda_0) f \in L_w^1(0, b)$, for some $\lambda_0 \in \mathbb{C}$ and $k = 1, \dots, n^2 N$. Setting

$$C_j(\lambda) = \sum_{j,k=1}^{n^2 N} \left|\xi^{jk}\right| \int_a^b \left|\overline{\varphi_k^+(t, \lambda_0)}\right| |f(s)| w(s) ds, \quad j = 1, 2, \dots, n^2 N. \tag{4.13}$$

Then

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n^2 N} \left(\left|\alpha_j(\lambda)\right| + C_j(\lambda)\right) \left|\varphi_j(t, \lambda_0)\right| + |\lambda - \lambda_0|$$

$$\times \sum_{j,k=1}^{n^2 N} \left|\xi^{jk}\right| \left|\varphi_j(t, \lambda_0)\right| \int_a^b \left|\overline{\varphi_k^+(t, \lambda_0)}\right| |f(s)| w(s) ds. \tag{4.14}$$

On application of the Cauchy-Schwartz inequality to the integral in (4.14), we get

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n^2 N} \left(\left|\alpha_j(\lambda)\right| + C_j(\lambda)\right) \left|\varphi_j(t, \lambda_0)\right| + |\lambda - \lambda_0| \sum_{j,k=1}^{n^2 N} \left|\xi^{jk}\right| \left|\varphi_j(t, \lambda_0)\right|$$

$$\times \left(\int_a^b \left|\overline{\varphi_k^+(t, \lambda_0)}\right|^2 w(s) ds\right)^{\frac{1}{2}} \left(\int_0^b |\varphi(s, \lambda)|^2 w(s) ds\right)^{\frac{1}{2}}. \tag{4.15}$$

From the inequality $(u + v)^2 \leq 2(u^2 + v^2)$, it follows that

$$|\varphi(t, \lambda)|^2 \leq 4 \sum_{j=1}^{n^2 N} \left(\left|\alpha_j(\lambda)\right| + C_j(\lambda)\right)^2 \left|\varphi_j(t, \lambda_0)\right|^2 + 4|\lambda - \lambda_0|^2 \sum_{j,k=1}^{n^2 N} \left|\xi^{jk}\right| \left|\varphi_j(t, \lambda_0)\right|^2$$

$$\times \left(\int_a^b \left|\overline{\varphi_k^+(t, \lambda_0)}\right|^2 w(s) ds\right) \left(\int_0^b |\varphi(s, \lambda)|^2 w(s) ds\right). \tag{4.16}$$

By hypothesis, there exist positive constant K_0 and K_1 such that

$$\left\|\varphi_j(t, \lambda_0)\right\|_{L_w^2(0,b)} \leq K_0 \text{ and } \left\|\overline{\varphi_k^+(s, \lambda_0)}\right\|_{L_w^2(0,b)} \leq K_1;$$

$$j, k = 1, 2, \dots, n^2 N. \tag{4.17}$$

Hence

$$|\varphi(t, \lambda)|^2 \leq 4\sum_{j=1}^{n^2N}\left(|\alpha_j(\lambda)| + C_j(\lambda)\right)^2|\varphi_j(t, \lambda_0)|^2 + 4K_1^2|\lambda - \lambda_0|^2$$

$$\times \sum_{j,k=1}^{n^2N}\left|\xi^{jk}\right|^2|\varphi_j(t, \lambda_0)|^2\left(\int_0^b|\varphi(s, \lambda)|^2 w(s)ds\right). \tag{4.18}$$

Integrating the inequality in (4.18) between 0 and t, we obtain

$$\int_0^t|\varphi(s, \lambda)|^2 w(s)ds \leq K_2 + \left(4|\lambda - \lambda_0|^2\sum_{j,k=1}^{n^2N}\left|\xi^{jk}\right|^2\right)$$

$$\times \int_0^t|\varphi_j(t, \lambda_0)|^2\left(\int_0^s|\varphi(x, \lambda)|^2 w(x)dx\right)w(s)ds, \tag{4.19}$$

where

$$K_2 = 4K_0^2\sum_{j=1}^{n^2N}\left(|\alpha_j(\lambda)| + C_j(\lambda)\right)^2. \tag{4.20}$$

Now, on using Gronwall's inequality, it follows that

$$\int_0^t|\varphi(s, \lambda)|^2 w(s)ds \leq K_2 \exp\left(4K_1^2|\lambda - \lambda_0|^2\sum_{j,k=1}^{n^2N}\left|\xi^{jk}\right|^2\int_0^t|\varphi_j(t, \lambda_0)|^2 w(s)ds\right). \tag{4.21}$$

Since, $\varphi_j(t, \lambda_0) \in L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and for $j = 1, \ldots, n^2N$, then $\varphi(t, \lambda) \in L_w^2(0, b)$.

Remark. Lemma 4.10 also holds if the function f is bounded on $[0, b)$.

Lemma 4.11. *Let* $f \in L_w^2(0, b)$. *Suppose for some* $\lambda_0 \in \mathbb{C}$ *that*

(i) *all solutions of* $(\prod_{j=1}^n \tau_j^+ - \overline{\lambda}I)\varphi^+ = 0$ *are in* $L_w^2(0, b)$.

(ii) $\varphi_j^{[r]}(t, \lambda_0)$, $j = 1, \ldots, n^2N$ *are bounded on* $[0, b)$ *for some* $r = 0, 1, \ldots, n^2N - 1$.

Then $\varphi^{[r]}(t, \lambda) \in L_w^2(0, b)$ *for any solution* $\varphi(t, \lambda)$ *of the equation* $[\prod_{j=1}^{n} \tau_j - \lambda I]\varphi = wf$ *for all* $\lambda \in \mathbb{C}$.

In the sequel, we shall require the following nonlinear integral inequality, which generalizes those integral inequalities used in [6, 7], [12], and [17-19].

Lemma 4.12 (cf. [1-4]). *Let* $u(t)$ *and* $v(t)$ *be two non-negative functions, locally integrable on the interval* $I = [0, b)$. *Then the inequality*

$$u(t) \leq c_0 + \int_0^t v(s) u^p(s) dx, \quad c_0 > 0,$$

for $0 \leq \sigma < 1$, *implies that*

$$u(t) \leq \left((c_0)^{(1-\sigma)} + (1 - \sigma) \int_0^t v(s) ds \right)^{\frac{1}{(1-\sigma)}} ds. \tag{4.22}$$

In particular, if $v(s) \in L^1(0, b)$, *then* (4.22) *implies that* $u(t)$ *is bounded.*

Lemma 4.13 (cf. [1-8]). *Let* $u(t)$, $z(t)$, $g(t)$, *and* $h(t)$ *be non-negative continuous functions defined on the interval* $I = [0, b)$ *and suppose that the inequality*

$$u(t) \leq z(t) + g(t) \left(\int_0^t u^2(s) h(s) dx \right)^{\frac{1}{2}} \text{ for } t \geq 0.$$

Then

$$u(t) \leq z(t) + g(t) \left(\int_0^t 2z^2(s) h(s) \exp\left[\int_0^s 2g^2(x) h(x) dx \right] ds \right)^{\frac{1}{2}} \text{ for } t \geq 0.$$

5. Boundedness Solutions of Product Equations

In this section, we shall consider the question of determining conditions under which all solutions of the Equation (1.3) are bounded and L_w^2-bounded.

Suppose there exist non-negative continuous functions $k(t)$ and $h_i(t)$ on $[0, b)$, $0 < b \leq \infty$; $i = 0, 1, \ldots, n^2N - 1$ such that

$$\left|F\left(t, y^{[0]}, y^{[1]}, \ldots, y^{[n^2N-1]}\right)\right| \leq k(t) + \sum_{i=0}^{n^2N-1} h_i(t)\left|y^{[i]}(t)\right|^{\sigma}, \text{ for } t \geq 0, \tag{5.1}$$

$-\infty < y^{[i]} < \infty$, for each $i = 0, 1, \ldots, n^2N - 1$ and for some $\sigma \in [0, 1]$; see [1], [8], and [18-19].

Theorem 5.1. *Suppose that F satisfies* (5.1) *with* $\sigma = 1$, $S^r(\tau) \cup S(\tau^+) \subset L^\infty(0, b)$ *for some* $r = 0, 1, \ldots, n^2N - 1$, *for some* $\lambda_0 \in \mathbb{C}$ *and that*

(i) $k(t) \in L_w^1(0, b)$ *for all* $t \in [0, b)$.

(ii) $h_i(t) \in L_w^1(0, b)$ *for all* $t \in [0, b)$, $i = 0, 1, \ldots, n^2N - 1$.

Then $\varphi^{[r]}(t, \lambda)$, $r = 0, 1, \ldots, n^2N - 1$ *are bounded on* $[0, b)$ *for any solutions* $\varphi(t, \lambda)$ *of the Equations* (1.3) *for all* $\lambda \in \mathbb{C}$.

Proof. Note that (5.1) and Lemma 4.6 implies that all solutions are defined on $[0, b)$; see [1], [2], [6, 7], and [13, Chapter 3]. Let $\{\varphi_1(t, \lambda_0), \varphi_2(t, \lambda_0), \ldots, \varphi_{n^2N}(t, \lambda_0)\}$, $\{\varphi_1^+(s, \lambda_0), \varphi_2^+(s, \lambda_0), \ldots, \varphi_{n^2N}^+(s, \lambda_0)\}$ be two sets of linearly independent solutions of the Equations (4.5), respectively, and let $\varphi(t, \lambda)$ be any solution of (1.3) on $[0, b)$, then by Lemma 4.7, we have

$$\varphi^{[r]}(t, \lambda) = \sum_{j=1}^{n^2N} \alpha_j(\lambda)\varphi_j^{[r]}(t, \lambda_0) + \frac{1}{i^{n^2N}}(\lambda - \lambda_0)\sum_{j,k=1}^{n^2N} \xi^{jk}\varphi_j^{[r]}(t, \lambda_0)$$

$$\times \int_a^t \overline{\varphi_k^+(s, \lambda_0)} F\left(s,\, y^{[0]},\, y^{[1]},\, \ldots,\, y^{[n^2N-1]}\right) w(s) ds,$$

$$\text{for } r = 0, 1, \ldots, n^2N - 1. \tag{5.2}$$

Hence

$$\left|\varphi^{[r]}(t, \lambda)\right| \le \sum_{j=1}^{n^2N} |\alpha_j(\lambda)| \left|\varphi_j^{[r]}(t, \lambda_0)\right| + |\lambda - \lambda_0| \sum_{j,k=1}^{n^2N} \left|\xi^{jk}\right| \left|\varphi_j^{[r]}(t, \lambda_0)\right| \int_a^t \left|\overline{\varphi_k^+(s, \lambda_0)}\right|$$

$$\times \int_0^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right| \left(k(s) + \sum_{i=0}^{n^2N-1} h_i(s)\left|\varphi^{[i]}(s, \lambda)\right|\right) w(s) ds,$$

$$r = 0, 1, \ldots, n^2N - 1. \tag{5.3}$$

Since $k(s) \in L_w^1(0, b)$ and $\varphi_k^+(s, \lambda_0)$, $k = 1, 2, \ldots, n^2N$ are bounded on $[0, b)$ for some $\lambda_0 \in \mathbb{C}$, we have $\varphi_k^+(s, \lambda_0)k(s) \in L_w^1(0, b)$, $k = 1, 2, \ldots, n^2N$ for some $\lambda_0 \in \mathbb{C}$. Setting

$$C_j = |\lambda - \lambda_0| \sum_{j,k=1}^{n^2N} \left|\xi^{jk}\right| \left|\int_a^t \overline{\varphi_k^+(t, \lambda_0)} k(s) w(s) ds, \; j = 1, 2, \ldots, n^2N. \tag{5.4}$$

Then

$$\left|\varphi^{[r]}(t, \lambda)\right| \le \sum_{j=1}^{n^2N} (C_j + |\alpha_j(\lambda)|)\left|\varphi_j^{[r]}(t, \lambda_0)\right| + |\lambda - \lambda_0| \sum_{j,k=1}^{n^2N} \sum_{i=1}^{n^2N-1} \left|\xi^{jk}\right| \left|\varphi_j^{[r]}(t, \lambda_0)\right|$$

$$\times \int_0^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right| h_i(s) \left|\varphi^{[i]}(s, \lambda)\right| w(s) ds, \; r = 0, 1, \ldots, n^2N - 1. \tag{5.5}$$

By hypothesis, there exist a positive constants K_0 and K_1 such that $\left|\varphi_j^{[r]}(t, \lambda_0)\right| \le K_0$ and $|\varphi_k^+(t, \lambda_0)| \le K_1$ for all $t \in [0, b)$, $j, k = 1, \ldots, n^2N$, $r = 0, 1, \ldots, n^2N - 1$. Hence, by summing both sides of (5.5) from $r = 0$ to $n^2N - 1$, we get

$$\sum_{r=1}^{n^2N-1}\left|\varphi^{[r]}(t,\lambda)\right| \le (n^2N-1)K_0\sum_{j=1}^{n^2N}\left(C_j+\left|\alpha_j(\lambda)\right|\right)+(n^2N-1)K_0K_1|\lambda-\lambda_0|$$

$$\times\sum_{j,k=1}^{n^2N}\left|\xi^{jk}\right|\int_0^t\left(\max_{0\le i\le n^2N-1}h_i(s)\right)$$

$$\times\left(\sum_{i=0}^{n^2N-1}\left|\varphi^{[i]}(s,\lambda)\right|\right)w(s)dx. \tag{5.6}$$

Applying Gronwall's inequality to (5.6) and using (ii), we deduce that $\sum_{r=1}^{n^2N-1}\left|\varphi^{[r]}(t,\lambda)\right|$ is finite and hence the result.

Remark. From [3, Section 3] and [4], φ and $\varphi^{[j]} \in L_w^1(0, b)$ implies that $\varphi^{[r]}(t,\lambda) \in L_w^1(0, b)$ for any solution $\varphi(t,\lambda)$ of the Equations (1.3) for all $\lambda \in \mathbb{C}$, $r = 1, \dots, j-1, 1 \le j \le n^2N-1$.

Theorem 5.2. *Suppose that F satisfies* (5.1) *with* $\sigma = 1$, $S^r(\tau)\cup S(\tau^+) \subset L_w^2(0, b)$, *for some* $\lambda_0 \in \mathbb{C}$ *and some* $r = 0, 1, \dots, n^2N-1$ *and that*

(i) $k(t) \in L_w^2(0, b)$ *for all* $t \in [0, b)$.

(ii) $h_i(t) \in L_w^2(0, b)$ *for all* $t \in [0, b)$, $i = 0, 1, \dots, n^2N-1$.

Then $\varphi^{[r]}(t,\lambda) \in L_w^2(0, b)$, $r = 0, 1, \dots, n^2N-1$ *for any solutions* $\varphi(t,\lambda)$ *of the Equations* (1.3) *for all* $\lambda \in \mathbb{C}$.

Proof. Applying the Cauchy-Schwartz inequality to the integral in (5.5), we get

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^{n^2N}\left(C_j+\left|\alpha_j(\lambda)\right|\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right|+|\lambda-\lambda_0|\sum_{j,k=1}^{n^2N}\sum_{i=0}^{n^2N-1}\left|\xi^{jk}\right|$$

$$\times\left|\varphi_j^{[r]}(t,\lambda_0)\right|\left(\int_0^t\left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2|h_i(s)|w(s)ds\right)^{\frac{1}{2}}$$

$$\times\left(\int_0^t|h_i(s)|\left|\varphi^{[i]}(s,\lambda)\right|^2w(s)ds\right)^{\frac{1}{2}}, \quad r = 0, 1, \dots, n^2N-1. \tag{5.7}$$

Since $\varphi_k^+(t, \lambda_0) \in L_w^2(0, b)$, for some $\lambda_0 \in \mathbb{C}$ and $h_i(t) \in L^\infty(0, b)$ by hypothesis, then $\varphi_k^+(t, \lambda_0)|h_i(t)|^{\frac{1}{2}} \in L_w^2(0, b)$, $k = 1, 2, \ldots, n^2N$, $i = 0, 1, \ldots, n^2N - 1$. Let

$$D_{ki} = \left(\int_0^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right|^2 |h_i(s)| w(s) ds\right)^{\frac{1}{2}}, \; z(t) = \sum_{j=1}^{n^2N} (C_j + |\alpha_j(\lambda)|) \left|\varphi_j^{[r]}(t, \lambda_0)\right|,$$

and

$$G(t) = |\lambda - \lambda_0| \sum_{j,k=1}^{n^2N} \sum_{i=0}^{n^2N-1} \left|\xi^{jk}\right| \left|\varphi_j^{[r]}(t, \lambda_0)\right|.$$

From Lemma 4.13, we have

$$\left|\varphi^{[r]}(t, \lambda)\right| \le Z(t) + G(t)\left(\int_0^t 2Z^2(s)|h_i(s)| \exp\left[\int_0^s 2G^2(x)|h_i(x)| w(x) dx\right] w(s) ds\right)^{\frac{1}{2}}.$$

Since $\int_0^t Z^2(s)|h_i(s)| w(s) ds$ and $\int_0^s G^2(x)|h_i(x)| w(x) dx$ are both finite, we conclude that $\varphi^{[r]}(t, \lambda)$ is bounded by a linear combination of $L_w^2(0, b)$ functions $Z(t)$ and $G(t)$. Therefore, by using Lemma 4.8, $\varphi^{[r]}(t, \lambda) \in L_w^2(0, b)$, $r = 0, 1, \ldots, n^2N - 1$ for all $\lambda \in \mathbb{C}$.

Remark. If we use the Cauchy-Schwartz inequality for the integral in (5.5) as

$$\int_0^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right| |h_i(s)| \left|\varphi^{[i]}(s, \lambda)\right| w(s) ds$$

$$\le \left(\int_0^t \left|\overline{\varphi_k^+(s, \lambda_0)}\right|^2 |h_i(s)|^2 w(s) ds\right)^{\frac{1}{2}} \left(\int_0^t \left|\varphi^{[i]}(s, \lambda)\right|^2 w(s) ds\right)^{\frac{1}{2}}, \; i = 1, 2, \ldots, n^2N - 1,$$

we also get the result. We refer to [1] and [2] for more details.

Corollary 5.3. *Suppose that* $\left|F\left(t,\ y^{[0]},\ y^{[1]},\ \dots,\ y^{[n^2N-1]}\right)\right| = \sum_{i=0}^{n^2N-1}$ $\times\ h_i(t)\left|y^{[i]}(t)\right|$, $S^r(\tau)\cup S(\tau^+)\subset L_w^2(0,\ b)$ *for some* $\lambda_0\in\mathbb{C}$ *and some* $i = 0, 1, \dots, n^2N-1$ *and that* $h_i(t)\in L_w^p(0,\ b)$ *for some* $p\geq 2,\ t\in[0,\ b)$; $i = 1, 2, \dots, n^2N-1$. *Then* $\varphi^{[r]}(t,\ \lambda)\in L_w^1(0,\ b)$ *for any solutions* $\varphi(t,\ \lambda)$ *of the Equations* (1.3) *for all* $\lambda\in\mathbb{C}$ *and all* $r = 0, 1, \dots, n^2N-1$.

Proof. The proof is similar to Theorem 5.2 and therefore omitted.

The special case $h_i(t) = 0,\ i = 0, 1, \dots, n^2N-1$ and $k(t)\in L_w^2(0,\ b)$ yields the result.

Corollary 5.4. *Suppose that for some* $\lambda_0\in\mathbb{C}$, *if all solutions of the equations* $[\prod_{j=1}^{n}\tau_j] = \lambda_0 wu$ *and* $[\prod_{j=1}^{n}\tau_j^+] = \overline{\lambda_0}wv$ *are in* $L_w^2(0,\ b)$ *for some* $\lambda_0\in\mathbb{C}$ *and* $k(t)\in L_w^2(0,\ b)$. *Then all solutions of the equations* $[\prod_{j=1}^{n}\tau_j - \lambda w]\varphi = wk$ *are in* $L_w^2(0,\ b)$ *for every complex number* $\lambda\in\mathbb{C}$.

Next, for considering (5.1) with $0\leq\sigma<1$, we have the following:

Theorem 5.5. *Suppose that* F *satisfies* (5.1) *with* $0\leq\sigma<1$, $S^r(\tau)\cup S(\tau^+)\subset L_w^2(0,\ b)$, *for some* $\lambda_0\in\mathbb{C}$ *and some* $r = 0, 1, \dots, n^2N-1$ *and that*

(i) $k(t)\in L_w^2(0,\ b)$ *for all* $t\in[0,\ b)$.

(ii) $h_i(t)\in L_w^{2/(1-\sigma)}(0,\ b)$ *for all* $t\in[0,\ b)$, $i = 0, 1, \dots, n^2N-1$.

Then $\varphi^{[r]}(t,\ \lambda)\in L_w^2(0,\ b)$, $r = 0, 1, \dots, n^2N-1$ *for any solutions* $\varphi(t,\ \lambda)$ *of the Equations* (1.3) *for all* $\lambda\in\mathbb{C}$.

Proof. For $0 \le \sigma < 1$, the proof is the same up to (5.5). In this case, (5.5) becomes

$$\left|\varphi^{[r]}(t, \lambda)\right| \le \sum_{j=1}^{n^2N}(C_j + |\alpha_j(\lambda)|)\left|\varphi_j^{[r]}(t, \lambda_0)\right| + |\lambda - \lambda_0| \sum_{j,k=1}^{n^2N}\sum_{i=1}^{n^2N-1}\left|\xi^{jk}\right|$$

$$\times \left|\varphi_j^{[r]}(t, \lambda_0)\right| \int_0^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right| h_i(s)\left|\varphi^{[i]}(s, \lambda)\right|^{\sigma} w(s)ds,$$

$$r = 0, 1, \dots, n^2N - 1. \tag{5.8}$$

Applying the Cauchy-Schwartz inequality to the integral in (5.8), we get

$$\int_0^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right| |h_i(s)| \left|\varphi^{[i]}(s, \lambda)\right|^{\sigma} w(s)ds$$

$$\le \left(\int_0^t \left|\overline{\varphi_k^+(s, \lambda_0)}\right|^2 |h_i(s)|^{\mu} w(s)ds\right)^{\frac{1}{\mu}} \left(\int_0^t \left|\varphi^{[i]}(s, \lambda)\right|^2 w(s)ds\right)^{\frac{\sigma}{2}}, \tag{5.9}$$

where $\mu = 2/(2-\sigma)$. Since $\varphi_k^+(t, \lambda_0) \in L_w^2(0, b)$ for some $\lambda_0 \in \mathbb{C}$, $k = 1, 2, \dots, n^2N$ and $h_i(s) \in L_w^{2/(1-\sigma)}(0, b)$ by hypothesis, then we have $\varphi_k^+(t, \lambda_0)|h_i(t)| \in L_w^{\mu}(0, b)$, for some $\lambda_0 \in \mathbb{C}$, $k = 1, 2, \dots, n^2N$; $i = 0, 1, \dots, n^2N - 1$. Using this fact and (5.9), we obtain

$$\left|\varphi^{[r]}(t, \lambda)\right| \le \sum_{j=1}^{n^2N}(C_j + |\alpha_j(\lambda)|)\left|\varphi_j^{[r]}(t, \lambda_0)\right| + K_0|\lambda - \lambda_0| \sum_{j,k=1}^{n^2N}\sum_{i=1}^{n^2N-1}\left|\xi^{jk}\right|$$

$$\times \left|\varphi_j^{[r]}(t, \lambda_0)\right| \left(\int_0^t \left|\varphi^{[i]}(s, \lambda)\right|^2 w(s)ds\right)^{\frac{\sigma}{2}},$$

$$r = 0, 1, \dots, n^2N - 1, \tag{5.10}$$

where $K_0 = \|\varphi_k^+(t, \lambda_0)h_i(t)\|_{\mu}$, $\|\cdot\|_{\mu}$ denotes the norm in $L_w^{\mu}(0, b)$. The inequality

$$(u + v)^2 \le 2(u^2 + v^2), \tag{5.11}$$

implies that

$$\left|\varphi^{[r]}(t, \lambda)\right|^2 \leq 4\sum_{j=1}^{n^2N}\left(C_j^2 + |\alpha_j(\lambda)|^2\right)\left|\varphi_j^{[r]}(t, \lambda_0)\right|^2 + 4K_0^2|\lambda - \lambda_0|^2$$

$$\times \sum_{j,k=1}^{n^2N}\sum_{i=1}^{n^2N-1}\left|\xi^{jk}\right|^2\left|\varphi_j^{[r]}(t, \lambda_0)\right|^2\left(\int_0^t\left|\varphi^{[i]}(s, \lambda)\right|^2 w(s)ds\right)^{\sigma},$$

$$r = 0, 1, \ldots, n^2N - 1. \tag{5.12}$$

Setting $K_1 = \int_0^t\left|\varphi_j^{[r]}(t, \lambda_0)\right|^2 w(s)ds$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \ldots, n^2N - 1;\ k = 1, \ldots, n^2N$ and integrating (5.12), we obtain

$$\int_0^t\left|\varphi^{[r]}(t, \lambda)\right|^2 w(s)ds \leq K_2 + 4K_0^2|\lambda - \lambda_0|^2\sum_{j,k=1}^{n^2N}\sum_{i=1}^{n^2N-1}\left|\xi^{jk}\right|^2\int_0^t\left|\varphi_j^{[r]}(s, \lambda_0)\right|^2$$

$$\times\left[\left(\int_0^s\left|\varphi^{[i]}(x, \lambda)\right|^2 w(x)dx\right)^{\sigma}\right]w(s)ds, \tag{5.13}$$

where $K_2 = 4\sum_{j=1}^{n^2N}\left(C_j^2 + |\alpha_j(\lambda)|^2\right)K_1$.

An application of lemma (4.12) for $0 \leq \sigma < 1$ and of Gronwall's inequality to (5.13) for $\sigma = 1$ yields the result.

Theorem 5.6. *Suppose that* F *satisfies* (5.1) *with* $0 \leq \sigma < 1$, $S^r(\tau) \cup S(\tau^+) \subset L_w^2(0, b) \cap L^\infty(0, b)$, *for some* $\lambda_0 \in \mathbb{C}$ *and some* $r = 0, 1, \ldots, n^2N - 1$ *and that*

(i) $k(t) \in L_w^2(0, b)$ *for all* $t \in [0, b)$.

(ii) $h_i(t) \in L_w^p(0, b)$ *for some* p, $1 \leq p \leq 2/(1-\sigma), i = 0, 1, \ldots, n^2N - 1$.

Then $\varphi^{[r]}(t, \lambda) \in L_w^2(0, b) \cap L^\infty(0, b)$, $r = 0, 1, \ldots, n^2N - 1$ *for any solutions* $\varphi(t, \lambda)$ *of the Equations* (1.3) *for all* $\lambda \in \mathbb{C}$.

Proof. Since $S^r(\tau) \cup S(\tau^+) \subset L_w^2(0, b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \ldots, n^2N - 1$, then $\varphi_j^{[r]}(s, \lambda_0), \varphi_k^+(t, \lambda_0) \in L_w^q(0, b), j, k = 0, 1, \ldots, n^2N$ for every $q \geq 2$ and for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \ldots, n^2N - 1$.

First, suppose that $h_i(t) \in L_w^p(0, b)$ for some $p, 1 \leq p \leq 2$. Setting

$$K_0 = \left\| \varphi_j^{[r]}(t, \lambda_0) \right\|_\infty \text{ and } K_1 = \|\varphi_k^+(t, \lambda_0)\|_\infty; \quad j, k = 0, 1, \ldots, n^2N,$$

for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \ldots, n^2N - 1$, we have from (5.8) that

$$\left|\varphi^{[r]}(t, \lambda)\right| \leq K_0 \sum_{j=1}^{n^2N} (C_j + |\alpha_j(\lambda)|) + K_0K_1|\lambda - \lambda_0|$$

$$\times \left(\sum_{j,k=1}^{n^2N} \sum_{i=1}^{n^2N-1} \left|\xi^{jk}\right| \left| \int_0^t h_i(s) \left|\varphi^{[i]}(s, \lambda)\right|^\sigma w(s) ds \right). \qquad (5.14)$$

Since $h_i(t) \in L_w^p(0, b)$ for some $p, 1 \leq p \leq 2$, then Lemma 4.12 together with Gronwall's inequality implies that $\varphi^{[r]}(t, \lambda) \in L^\infty(0, b)$ for all $\lambda \in \mathbb{C}$, i.e., there exists a positive constant K_3 such that

$$\left|\varphi^{[r]}(t, \lambda)\right| \leq K_3 \text{ for all } \lambda \in \mathbb{C}, \quad t \in [0, b), r = 0, 1, \ldots, n^2N - 1. \quad (5.15)$$

From (5.8) and (5.15), we obtain

$$\left|\varphi^{[r]}(t, \lambda)\right| \leq \sum_{j=1}^{n^2N} \left(C_j + |\alpha_j(\lambda)| + K_3\right) \left|\varphi_j^{[r]}(t, \lambda_0)\right|,$$

for any appropriate constant K_3. Since $\varphi_j^{[r]}(t, \lambda_0) \in L_w^2(0, b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \ldots, n^2N - 1$, this proves $\varphi^{[r]}(t, \lambda) \in L_w^p(0, b)$ for all $\lambda \in \mathbb{C}, 1 \leq p \leq 2$.

Next, suppose that $h_i(t) \in L_w^p(0, b)$ for some $p, 2 < p \leq 2/(1 - \sigma)$, $i = 0, 1, \ldots, n^2N - 1$. Define $q \geq 2$ by

$$\frac{1}{q} = \frac{2-\sigma}{2} - \frac{1}{p}$$

(which is possible because of the restriction on p). Thus $\varphi_j^{[r]}(s, \lambda_0)$, $\varphi_k^{+}(t, \lambda_0) \in L_w^q(0, b)$ and $\varphi_k^{+}(t, \lambda_0)h_i(t) \in L_w^{\mu}(0, b)$, $\mu = 2/(2-\sigma)$.

Repeating the same argument in the proof of Theorem 5.5 and from (5.9) to (5.13), we obtain that $\varphi^{[r]}(t, \lambda) \in L_w^2(0, b)$. Returning to (5.9), we find that the integral on the left-hand side is bounded, which implies, by (5.8) that

$$\left|\varphi^{[r]}(t, \lambda)\right| \leq \sum_{j=1}^{n^2 N} \left(C_j + \left|\alpha_j(\lambda)\right| + K_3\right)\left|\varphi_j^{[r]}(t, \lambda_0)\right|,$$

for an appropriate constant K_3. Since $\varphi_j^{[r]}(t, \lambda_0) \in L^{\infty}(0, b)$, this completes the proof . We refer to [1], [3], and [6, 7] for more details.

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